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Scalar-Vector Topological Soliton

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We present classical scalar-vector equations which admit soliton solutions in three space dimensions. Exact spherical solutions are obtained which are everywhere regular and resemble charged particles of finite self-energy. The corresponding 4-current is identically conserved and leads to quantized charges. The scale of the soliton is unique and determined by boundary conditions, which also ensure its topological stability.

1. INTRODUCTION

It is well known that for a scalar field theory in more than one spatial dimension, time-independent soliton solutions do not exist. This is Derrick's (1964) theorem. Accordingly, the interesting search for soliton solutions in real 3D space should either employ multidimensional, time-dependent solutions or fields of nonzero spin (see, e.g., Rajaraman, 1982; Guidry, 1991). Soliton solutions are usually classified into two groups: topological and nontopological. Topological solitons are those which owe their stability to nontrivial boundary conditions and topology, while the stability of nontopological solitons is dynamical and is not associated with boundary conditions. Vortex lines of Nielsen and Olesen (1973) and magnetic monopoles of 't Hooft (1974) and Polyakov (1974) are examples of topological solitons. For the nontopological solitons the reader is referred to Lee (1981) and Rajaraman (1982).

In this paper, we introduce scalar-vector field equations which are designed to meet the following properties:

1. Relativistic covariance.
2. Existence and conservation of a 4-current.

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3. Existence of singularity-free, $3D$ topological solitons with quantized charges.
4. Inverse square field intensity of the charged soliton at large distances.
5. Existence of antisolitons with opposite charges.

The conserved current is not a Noether current and does not emerge as a result of an underlying symmetry. Despite its resemblance to some of the work on properties of charged particles, the present work is to be considered as an exercise in mathematical physics and not as a physical theory.

2. FIELD EQUATIONS

We start from the relativistic equations

$$\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c} J^\beta \tag{1}$$

$$\Box^2 \phi = \frac{\partial V}{\partial \phi} \tag{2}$$

in which

$$G^{\alpha\beta} = \partial^\alpha B^\beta - \partial^\beta B^\alpha \tag{3}$$

where B^α ($\alpha = 0, 1, 2, 3$) is a vector field, and

$$\phi = (-\phi^a \phi_a)^{1/2} \tag{4}$$

where ϕ^a ($a = 1, 2, 3$) is an isovector or a three-component scalar field (i.e., each component is invariant under Lorenz transformations). The potential $V(\phi)$ is chosen as

$$V(\phi) = \lambda(\phi - \phi_0)^6 \tag{5}$$

and the four-current J^β is defined according to

$$J^\beta = \kappa \epsilon^{\ \ \beta}_{\alpha\delta\gamma} \epsilon_{abc} \partial^\alpha \phi^a \partial^\delta \phi^b \partial^\gamma \phi^c \tag{6}$$

In equations (5) and (6), λ, ϕ_0, and κ are all real, positive constants, and $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor ($\epsilon^{abc} = \epsilon^{0abc}$). J^β is easily shown to be conserved identically

$$\partial_\beta J^\beta = 0 \tag{7}$$

We could adopt the equation $\Box^2 \phi^a = \partial V / \partial \phi_a$ instead of (2), but this would prevent us from the simple analytical treatment of the next section.

Our choice of the metric tensor is $g_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$ throughout this paper.

We also define $J^{\alpha} = (c\rho, \mathbf{J})$ and $B^{\alpha} = (\psi, \mathbf{A})$.

With the definition (3) for $G^{\alpha\beta}$, the following homogeneous equation is also satisfied identically:

$$\partial_{\alpha}\tilde{G}^{\alpha\beta} = 0 \tag{8}$$

where $\tilde{G}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}G_{\gamma\delta}$ is the dual tensor. Note that in (5), the minimum of V occurs at $\phi = \phi_0 > 0$, causing spontaneous breakdown of the $O(3)$ symmetry of the ϕ^a-field. Equation (2) is a dynamical equation which governs the magnitude of the ϕ^a-field and not its direction.

For soliton solutions which are localized field configurations of finite energy, we should obviously have $\phi \to \phi_0$ in the background region (i.e., $r \to \infty$). The total charge of the configuration is quantized (Arafune *et al.*, 1975)

$$\begin{aligned} Q = \frac{1}{c}\int J^0\, d^3x &= \frac{\kappa}{c}\int \epsilon^{0}_{\alpha\gamma\delta}\epsilon_{abc}\partial^{\alpha}\phi^a\partial^{\gamma}\phi^b\partial^{\delta}\phi^c\, d^3x \\ &= \frac{\kappa}{c}\int \epsilon^{0}_{\alpha\gamma\delta}\epsilon_{abc}\partial^{\alpha}[\phi^a\partial^{\gamma}\phi^b\partial^{\delta}\phi^c]\, d^3x \\ &= \frac{\kappa}{c}\int \epsilon_{ijk}\epsilon_{abc}\phi^a\partial^j\phi^b\partial^k\phi^c\, d\sigma^i \\ &= \frac{2\kappa\phi_0^3}{c}\int d\Omega = ne \end{aligned} \tag{9}$$

where $e = 8\pi k\phi_0^3/c$ is defined as the fundamental charge. Here, $d\sigma^i$ is the surface element in the x^i direction, which is integrated over the surface area of a coordinate sphere S^2 at infinity, while $d\Omega$ is the element of solid angle in the internal ϕ^a space. n is called the winding number (or Brouwer degree of mapping) and is interpreted topologically as the number of times that the vector ϕ^a sweeps the sphere of radius $\phi = \phi_0$ in the (ϕ^1, ϕ^2, ϕ^3) space as the radius vector $\mathbf{r}$ sweeps the large sphere of radius $r = R \to \infty$ in the coordinate space (x^1, x^2, x^3). In other words, there is a mapping

$$S^2(\text{coordinate space}) \to S^2(\text{field space}) \tag{10}$$

which forms a second homotopy group $\pi_2(S^2) = Z$.

3. SPHERICAL SOLITONS

In this section, we present spherically symmetric, soliton solutions of equations (1) and (2) in the form $B^{\alpha} = (\psi(r), \mathbf{0})$ and $\phi^a = \phi(r)r^a/r$. Using this ansatz in equations (1), (2), and (6), we get (after straightforward calculation)

$$J^0 = c\rho = 6\kappa \frac{\phi'(r)\phi^2(r)}{r^2} \tag{11}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi(r)}{dr}\right) = -\frac{4\pi}{c} J^0 \tag{12}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = -6\lambda(\phi - \phi_0)^5 \tag{13}$$

With slight modifications, equation (13) is Emden's equation (Weinberg, 1972) with the polytropic index $n = 5$, or $\gamma = 6/5$. Let us define $E(r) = -d\psi/dr$. Then the following solutions satisfy equations (13) and (12):

$$\phi(r) = \phi_0\left[1 - \left(1 + \frac{r^2}{r_0^2}\right)^{-1/2}\right] \tag{14}$$

$$E(r) = \frac{8\pi\kappa}{c}\frac{\phi^3(r)}{r^2} = \frac{e}{r^2}\left(\frac{\phi}{\phi_0}\right)^3 \tag{15}$$

Inserting (14) into (13), we find that this equation is satisfied if

$$r_0^2 = \frac{1}{2\lambda\phi_0^4} \tag{16}$$

Equation (13) has other solutions with arbitrary r_0, but (14) is the only solution which gives $\phi(r = 0) = 0$. ϕ should vanish at $r = 0$ if the ϕ^a is to be single-valued.

Solutions (14) and (15) correspond to $n = 1$ in (9). The total charge can also be calculated directly from

$$Q = \int \rho\, d^3x = \frac{6\kappa}{c}\int_0^\infty \frac{\phi'\phi^2}{r^2}\, 4\pi r^2\, dr = \frac{8\pi\kappa\phi_0^3}{c} = e \tag{17}$$

From (14) and (15), the configuration behaves like a point charge e, and $E(r)$ obeys an almost inverse square law. The asymptotic behavior of the soliton solution is as follows:

$$\phi(r) \simeq \phi_0\left(1 - \frac{r_0}{r}\right)$$

$$E(r) \simeq \frac{e}{r^2} \tag{18}$$

for $r \gg r_0$, and

$$\phi(r) \simeq \frac{1}{2}\phi_0\left(\frac{r}{r_0}\right)^2$$
$$E(r) \simeq \frac{e}{8r_0^2}\left(\frac{r}{r_0}\right)^4 \tag{19}$$

for $r \ll r_0$. The configuration is everywhere nonsingular.

The moving soliton solutions can be easily obtained from the static solutions (14) and (15) by applying a Lorentz boost.

4. ENERGY CONSIDERATIONS AND STABILITY

We did not derive the basic equations (1) and (2) from a Lagrangian density through the variational principle. We cannot, therefore, calculate formally the complete energy-momentum tensor of the fields concerned. However, we tentatively consider the following energy terms:

$$\mathscr{E} = \mathscr{E}_1 + \mathscr{E}_2 + \mathscr{E}_3 \tag{20}$$

where

$$\mathscr{E}_1 = \int_0^\infty \frac{1}{8\pi} E^2(r) 4\pi r^2\, dr = \frac{I_1 e^2}{r_0} \tag{21}$$
$$\mathscr{E}_2 = \int_0^\infty \frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 4\pi r^2\, dr = I_2 \phi_0^2 r_0 \tag{22}$$

and

$$\mathscr{E}_3 = \int_0^\infty \lambda(\phi - \phi_0)^6 4\pi r^2\, dr = I_3 \lambda \phi_0^6 r_0^3 = \frac{1}{2} I_3 \phi_0^2 r_0 \tag{23}$$

where I_1, I_2, and I_3 are three definite integrals

$$I_1 = \frac{1}{2}\int_0^\infty \frac{[1-(1+x^2)^{-1/2}]^6}{x^2}\, dx = 31 - \frac{315}{32}\pi$$
$$I_2 = 2\pi \int_0^\infty \frac{x^4\, dx}{(1+x^2)^3} = \frac{3\pi^2}{8}$$
$$I_3 = 4\pi \int_0^\infty \frac{x^2\, dx}{(1+x^2)^3} = \frac{\pi^2}{4} \tag{24}$$

The total energy thus obtained is obviously positive definite and finite. As a function of r_0, $\mathscr{E}(r_0)$ has an absolute minimum at

$$r_{00} = \left[\frac{(I_2^2 + 12 I_1 I_3 \lambda \phi_0^2 e^2)^{1/2} - I_2}{6 I_3 \lambda \phi_0^4}\right]^{1/2} \tag{25}$$

r_{00} is always greater than zero and does not necessarily coincide with the previously determined r_0 of (16) unless there is a fine-tuning relationship among the three parameters ϕ_0, λ, and e:

$$\lambda\phi_0^2 e^2 = \frac{2}{3}\frac{I_2}{I_1} \tag{26}$$

From a topological point of view, the stability of the configuration is due to the fact that the boundary condition $\phi^a(r \to \infty) \to \phi_0 r^a/r$ cannot be continuously deformed to a trivial boundary condition like $\phi^a \to \mathbf{C}$, where $\mathbf{C}$ is a constant vector in the (ϕ^1, ϕ^2, ϕ^3) space.

5. CONCLUSION

We have presented scalar-vector field equations which admit soliton solutions with quantized topological charges. Spherically symmetric, static solutions corresponding to $n = 1$ were derived analytically and the scale of the configuration was shown to be determined by boundary conditions at $r = 0$. The soliton resembles a singularity-free charged particle of finite mass.

In principle, by relaxing the spherical symmetry assumption and letting $\phi(r, \theta)$, $\mathbf{A}(r, \theta)$, and $\psi(r, \theta)$ depend on θ, hypothetical models of charged particles possessing spin can be constructed.

Obviously, the transformation $\phi^a \to -\phi^a$ leads to $Q \to -Q$, $\phi_{(r)} \to \phi_{(r)}$, $E_{(r)} \to -E_{(r)}$, and $\mathscr{E} \to \mathscr{E}$. This means that for each soliton solution there is a corresponding antisoliton solution with the same energy but opposite charge. The plane wave ansatz

$$\phi^a = \hat{\phi}^a \phi_0 e^{ik_\mu x^\mu} \qquad \text{and} \qquad B^\alpha = B_0^\alpha e^{ik_\mu x^\mu}$$

yields

$$J^\beta = -i\kappa\epsilon^\beta_{\alpha\delta\gamma}\epsilon_{abc}k^\alpha k^\delta k^\gamma \phi^a\phi^b\phi^c = 0 \qquad \text{and} \qquad k_\alpha \beta_0^\alpha = k_\alpha k^\alpha = 0$$

Therefore, homogeneous wave equations like those in classical electrodynamics could be obtained for the components of $G^{\alpha\beta}$. This shows that the scalar-vector waves propagate with the speed of light in vacuum.

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